

Geocentrism re-examined

Jeremy Goodman*

Princeton University Observatory, Peyton Hall, Princeton, NJ 08544

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Abstract

Observations show that the universe is nearly isotropic on very large scales. It is much more difficult to show that the universe is radially homogeneous—that is, independent of distance from us—or equivalently, that the universe is isotropic about distant points. This is usually taken as an axiom, since otherwise we would occupy a special position. Here we consider several empirical arguments for radial homogeneity, all of them based on the cosmic microwave background (CMB). We assume that physical laws are uniform, but we suppose that structure on very large scales may not be. The tightest limits for inhomogeneity on the scale of the horizon appear to be of order ten percent. These involve observations of the Sunyaev-Zel'dovich effect in clusters of galaxies, excitation of low-energy atomic transitions, and the accurately thermal spectrum of the CMB. Weaker limits from primordial nucleosynthesis are discussed briefly.

98.65, 98.80, 95.10, 95.30

I. INTRODUCTION

Homogeneity and isotropy are independent cosmological assumptions. General relativity allows homogeneous but anisotropic universes (*e.g.* [1]), and also spherically symmetric but inhomogeneous ones (*e.g.* [2]). If the universe is isotropic around two or more distinct points, however, then it must be homogeneous.

Homogeneity is more fundamental and powerful than isotropy but also more difficult to verify. Homogeneity allows local measurements to be applied to the whole universe; and conversely it allows observations of high-redshift regions to constrain the history of the local volume. Without homogeneity, modern cosmology would be very difficult. Nevertheless, inhomogeneous models are occasionally proposed. For example, it has been suggested recently that spherically symmetric, inhomogeneous universes are a natural consequence of inflation [3]. Such speculations, though unorthodox, demonstrate that homogeneity is not yet fully established.

By contrast, the *isotropy* of the universe on large scales *is* well established. It is supported by deep, wide-angle surveys of radio [4] and infrared [5] sources. Results from the Cosmic Background Explorer satellite (COBE) show that the temperature of the microwave background (CMB) deviates slightly from isotropy, but only at the level $(\Delta T/T)_{\text{rms}} \approx 1.1 \times 10^{-5}$ on angular scales $\geq 10^\circ$, apart from a dipole pattern that is conventionally attributed to the peculiar velocity of the Sun and the Galaxy [6].

To the extent that the universe is isotropic, it can be inhomogeneous only if it is symmetric around ourselves. We therefore ask whether present observations permit large-scale *radial* inhomogeneity, and if so, what future measurements might detect or exclude it.

Galaxy counts against redshift or magnitude are consistent with a homogeneous, uniformly-populated universe in the redshift range $0.03 \leq z \leq 0.3$, although statistical fluctuations associated with structure on scales $\leq 30h^{-1}\text{Mpc}$, due perhaps to the relatively narrow fields surveyed, make strong limits difficult to obtain [7]. By $z \approx 0.4$, counts in the blue are already discrepant, perhaps because of rapid evolution among fainter galaxies [8]. Even conservative models predict significant evolutionary effects on near-infrared counts by $z \sim 1$, and the model parameters, though easily adjusted to fit the data, are poorly constrained *a priori* [9]. This is typical of the evolutionary uncertainties that for decades have prevented the use of galaxies and other beacons to determine cosmic geometry, even when homogeneity is assumed, because noneuclidean effects are strong only for objects so distant as to be seen when the universe was much younger than it is now.

There may exist “standard candles” at $z \gtrsim 1$, such as Type I supernovae [10]. Among homogeneous Friedmann models, unfortunately, the shape of the magnitude-redshift relation for standard candles already depends on two parameters: the density parameter, Ω , and the cosmological constant, Λ . Only superb data will permit one to fit for a third parameter and thereby constrain the homogeneity of the universe on the scale of the present horizon. Similar remarks apply to more recently-proposed cosmological tests, such as the use of gravitational lenses to determine the dependence of angular-diameter distance on redshift [11].

The prospects for constraining the homogeneity of the CMB are better. In Sec. II we discuss two observational tests that are sensitive to radial inhomogeneity of the CMB in first order. Both of these involve measurement of the angle-averaged temperature of the CMB seen by a distant object, either through scattering or molecular absorption. Using

recent measurements, we can limit generic radial inhomogeneities to $\lesssim 10\%$. We then show in Sec. III that if one assumes a substantial fraction of all baryons to reside in an ionized intergalactic medium, then the accurately thermal nature of the CMB spectrum provides another $\sim 10\%$ limit, due to second-order effects of scattering. In Sec. III we compare these limits with an argument for homogeneity based on light-element nucleosynthesis.

II. THERMOMETERS AT MODERATE REDSHIFT

The CMB is homogeneous if it is isotropic around distinct points. Imagine therefore that one is provided with a mirror at a cosmological distance, and that the mirror is tilted at some angle to the line of sight. If the universe is isotropic around the distant mirror and the mirror has negligible peculiar velocity, then the CMB spectrum seen in the mirror is the same as that seen directly along unobstructed lines of sight. If the universe is not homogeneous, then it cannot be isotropic around both us and the mirror, so the mirror spectrum will generally differ from the direct spectrum.

To elaborate this idea, imagine that the mirror is half silvered, so that it reflects a fraction f of the radiation and transmits the rest (this ideal mirror does not absorb). Then the spectrum seen in the mirror is

$$\begin{aligned} I_\nu &= (1 - f)B_\nu(T_0) + fB_\nu(T_r) \\ &\approx B_\nu[(1 - f)T_0 + fT_r] + O(\Delta T^2), \end{aligned} \quad (1)$$

where $B_\nu(T_0)$ and $B_\nu(T_r)$ are the direct and reflected spectra. We have assumed that the spectrum in any single direction is thermal. The combined spectrum is not, unless $T_r = T_0$, but it can be approximated by a thermal spectrum to first order in $T_r - T_0$.

Electron scattering serves as such a mirror. One requires a cluster of galaxies at redshift $z_{\text{cl}} \sim 1$ with a nonnegligible electron-scattering optical depth, τ . If the cluster fills the telescope beam, the observed spectrum summed over polarizations is, for $\tau \ll 1$,

$$\begin{aligned} I_\nu^{\text{obs}} &= (1 - \tau)B_\nu(T_0) + \tau \int \frac{3}{4}(1 + \cos^2 \psi) I'_\nu(\Omega) \frac{d^2\Omega}{4\pi} \\ &\quad + y \nu^4 \frac{\partial}{\partial \nu} \frac{1}{\nu^2} \frac{\partial}{\partial \nu} B_\nu(T_0), \end{aligned} \quad (2)$$

where $B_\nu(T_0)$ is the unscattered thermal spectrum, obtained from other lines of sight; $(1 + z_{\text{cl}})^3 I'_\nu(\Omega)$ is the specific intensity at the cluster in the direction Ω ; and ψ is the scattering angle between this direction and the line of sight. The factor of $(1 + \cos^2 \psi)$ expresses the angular dependence of electron scattering, summed over polarizations. Similar formulae hold for the individual polarizations, with integrands depending differently on the scattering angles. The third term on the right of Eq. (2) is the Sunyaev-Zel'dovich distortion due to the finite temperature of the electrons ($T_e \gg T_0$): $y \equiv \tau k_B T_e / m_e c^2$ [12]. Since the first and last term have a known dependence on frequency, multifrequency observations can be used to constrain the middle term.

Even in a homogeneous universe, a radial peculiar velocity v_r produces a distortion equivalent to the middle term in Eq. (2). To first order, the anisotropic part of $I'_\nu(\Omega)$ is then

a dipole pattern of amplitude v_r/c relative to the monopole [13]. The angular average of this dipole is zero in the cluster rest frame, so that the cluster sees the same average temperature as it would if $v_r = 0$. However, an observer at rest with respect to the CMB who views the cluster along direction \hat{n} sees the scattered photons to have suffered a decrease in their energy by a factor of $(1 - \vec{v} \cdot \hat{n}/c)$ on average. Thus, the conventional interpretation of any non- y -type distortion would be that the cluster has a peculiar velocity. In a homogeneous universe, the sign of v_r should be random on very large scales. In a radially inhomogeneous universe, v_r will have a trend with systematically with redshift, and $\langle v_r \rangle \rightarrow 0$ as $z \rightarrow 0$ because of spherical symmetry. Sec.III calculates these effects explicitly in linear theory for a universe close to an Einstein-de Sitter model.

The Sunyaev-Zel'dovich effect has now been measured in several clusters out to $z \sim 0.2$ in the Rayleigh-Jeans part of the spectrum, where positive peculiar velocities can not be distinguished from the y distortion [14]. Successful results near the Wien peak have been reported recently [15]. The temperature decrements are 1-2 mK with one-sigma errors ~ 10 -30%. No *increments* have been reported, such as might be produced by a large negative v_r .

Lacking multifrequency data for individual clusters, we may derive a limit on v_r and hence on radial inhomogeneity by the following argument. Assuming homogeneity and neglecting v_r , several groups have combined SZ measurements with X-ray data to derive Hubble's constant, H_0 [16]. The result scales as $H_0 \propto \Sigma_X T_e^{5/2}/y^2$, where Σ_X is the X-ray surface brightness at energies $< k_B T_e$. The electron temperature, T_e , can be estimated directly from the X-ray spectrum. Although the results are smaller than some local estimates of H_0 [17], they fall within a factor ~ 2 , which indicates that peculiar velocities have not altered the estimates of y by more than $\sim \sqrt{2}$. Hence

$$\frac{y_{\text{true}}}{\sqrt{2}} < y_{\text{true}} + \frac{v_r \tau}{2c} < \sqrt{2} y_{\text{true}}, \quad (3)$$

$$-0.012 < \frac{v_r}{c} < 0.016, \quad (4)$$

since $y_{\text{true}}/\tau = k_B T_e/m_e c^2 \approx 0.02$ for a typical temperature of 10 keV. In a radially inhomogeneous universe smooth on sufficiently large scales, the mean value of v_r would vary linearly with z at small z . Hence we should divide the above limit on v_r/c by the typical cluster redshift $z \approx 0.2$ to obtain a limit on inhomogeneity $\approx 8\%$. Since the temperature decrement is independent of distance, multifrequency measurements at higher z could—and probably soon will—improve this limit substantially.

Peculiar velocities and inhomogeneities can also be constrained by using atomic and molecular excitation as a thermometer for the CMB [18]. One measures optical absorption from an excited level lying $\sim k_B T_{\text{CMB}}$ above the ground state. Clearly, it is important to use systems in which collisional excitation is small or negligible, so that the observed excitation represents the angle-averaged radiation temperature seen by the atomic or molecular system, $T_{\text{CMB},z}$. Any discrepancy between this temperature and the redshifted temperature of the local CMB can be explained by a radial peculiar velocity:

$$\frac{v_r}{c} \approx \frac{(1+z)T_{\text{CMB},0}}{T_{\text{CMB},z}} - 1 \quad (5)$$

If v_r is large or if it has a trend with z , one has evidence of radial inhomogeneity.

Fine-structure lines of neutral carbon have been measured in absorption (against a background quasar) at $z = 1.776$, with the result that $\bar{T}_{\text{CMB},z} = 7.4 \pm 0.8$ K, as compared to $(1+z)T_{\text{CMB},0} = 7.58$ K [19]. Since the agreement is well within the errors, we can use the errors to set a limit $\sim 0.8/7.58 = 11\%$.

Whether we consider scattering or absorption, the tests of this section are sensitive primarily to dipole¹ anisotropies in the CMB as it might be seen by distant observers. For a cluster or absorption system at a given redshift z , the measured temperature difference is linearly proportional to the strength of the dipole seen at z . The test described in the next section is sensitive to distant anisotropies of all multipoles, but at second rather than first order.

III. SPECTRAL DISTORTIONS BY DIFFUSE SCATTERING

Sufficiently strong radial inhomogeneity at $z \sim 1000$ would produce a noticeable spectral distortion because of the finite thickness of the recombination surface [20]. This would measure inhomogeneities on a comoving scale only $\sim 10^{-2}$ of the present horizon.

A spectral distortion sensitive to much larger scales could arise from scattering by plasma associated with Ly α clouds and a possible intercloud medium. Absorption lines in quasar spectra reveal the presence of diffuse, probably intergalactic, clouds at $z \sim 2 - 4$ containing small amounts of atomic hydrogen. Physical considerations indicate that the hydrogen must be predominantly ionized, and it is plausible that the ionized intergalactic medium contains a significant fraction $f_{\text{IGM}} \sim 1$ of the hydrogen indicated by cosmic nucleosynthesis arguments [20,21], namely [22]

$$\bar{n}_H \approx 0.76\bar{n}_B \approx (1.1 \pm 0.2) \times 10^{-7} \text{cm}^{-3}. \quad (6)$$

As discussed in Sec. (IV), the luminous parts of galaxies account for only a fraction of the \bar{n}_H cited above, so that most of the baryons in the universe must be sequestered in some form other than visible stars. Although an ionized IGM is not the only possible hiding place for these baryons, it is a plausible one because hot gas accounts for most the baryonic mass in X-ray-emitting clusters of galaxies [23,24], and because searches sensitive to local neutral hydrogen have found it in amounts much smaller than Eq. (6) [25].

If the ionized IGM has persisted from $z_{\text{ion}} = 4$ to the present, then its total optical depth is $\tau_{\text{IGM}} \approx 10^{-2} f_{\text{IGM}}$ if the present age of the universe is 13Gyr. (With this choice of z_{ion} , τ_{IGM} is almost independent of Ω for $\Lambda = 0$.) On this assumption, about 1% of the CMB photons that we observe in any direction have been scattered at least once. To the extent that the electrons are cold ($k_B T_e \ll m_e c^2$), these scatterings have negligible effect on the CMB spectrum in a homogeneous universe, but they will produce a slightly nonthermal spectrum in a radially inhomogeneous universe.

¹In fact, by analyzing the two photon polarizations separately in the cluster test, one could measure both the dipole and one linear combination of quadrupole moments.

To make this discussion more quantitative, we consider scattering in a spherically-symmetric but radially inhomogeneous matter-dominated universe [2]. Since we cannot consider all possible forms of inhomogeneity, we shall adopt the following simplifying assumptions and hope that our results are representative of more difficult cases:

- (S.1) The model departs only slightly from a homogeneous matter-dominated Einstein-de Sitter universe, so that we may use first-order perturbation theory.
- (S.2) The inhomogeneities are growing adiabatic perturbations.
- (S.3) Recombination occurs instantaneously at a fixed value of the local temperature $T(r, t) = T_{\text{rec}}$.
- (S.4) After recombination, the universe is sufficiently optically thin that CMB photons scatter at most once before reaching us, and most photons do not scatter at all.
- (S.5) The photon energy is not changed in the local matter rest frame by the scattering process.

Assumptions (S.1) and (S.2) are arbitrary and could be relaxed, but they permit an easy Sachs-Wolfe treatment. Concerning (S.3), the actual width of the recombination epoch has been calculated to be $\Delta t/t \approx 0.2$ [20]. but to treat this epoch properly would require a dynamical analysis of the interaction between matter and inhomogeneous radiation. Assumption (S.4) is probably justified. The possibility that the universe was reionized early enough to produce a substantial optical depth has been much discussed, but this appears unlikely because CMB anisotropies are now seen on degree scales [26]. As we shall see, the spectral distortions produced by inhomogeneity would be indistinguishable from those caused by a hot intergalactic medium in a homogeneous universe. The two effects are additive and cannot be made to cancel, so assumption (S.5) is conservative.

We look out towards the recombination epoch along past-directed null geodesics. According to assumption (S.4), most of the CMB photons we receive never scattered on their way to us, so these photons sample the recombination epoch on a sphere of comoving radius r_d . (“d” for “direct”.) Because of isotropy, all photons reaching us from this sphere have been drawn from a Planck distribution with a common temperature and have suffered the same redshift, so their spectrum is completely thermal.

A minority of photons have scattered once. Consider all such photons that have scattered towards us through angle ψ at some epoch t_s , where $t_{\text{rec}} < t_s < t_{\text{now}}$. Traced backwards from t_s , the paths of these photons intercept the recombination era on a common sphere of radius $r_i(t_s, \psi) < r_d$ (“i” for “indirect”). Because of assumption (S.2), photons from the spheres r_d and r_i were drawn from the same Planck distribution. In a homogeneous universe, photons from both spheres would suffer the same redshift, $(1 + z_{\text{rec}})$, and because of this and assumption (S.5), the CMB spectrum would be unaffected by scattering. In an inhomogeneous universe, however, there is a first-order perturbation in these redshifts:

$$\zeta \equiv \delta \ln(1 + z) = \frac{1}{3} \{ \phi(0) - \phi[r(\eta_{\text{rec}})] \} + \frac{1}{3} \left\{ (1 - \cos \psi) \eta_s \frac{d\phi}{dr} [r(\eta_s)] + \eta_{\text{rec}} \left(\frac{dr}{d\eta} \frac{d\phi}{dr} \right)_{r(\eta_{\text{rec}})} \right\} \quad (7)$$

Here $r = \sqrt{x^2 + y^2 + z^2}$ is the comoving radius in the unperturbed Einstein-de Sitter metric,

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2), \quad (8)$$

η is the arc parameter,

$$\eta(t) \equiv \int_0^t \frac{dt'}{a(t')}, \quad (9)$$

$\phi(r)$ is the potential fluctuation associated with the perturbation in the mass density,

$$\phi(\vec{r}) = -a^2(t)G \int \frac{d^3\vec{r}'}{|\vec{r} - \vec{r}'|} \delta\rho(\vec{r}', t), \quad (10)$$

and the path of the photon is (for $\eta_{\text{rec}} \leq \eta_s \leq \eta_0$)

$$r(\eta) = \begin{cases} [(\eta_0 - \eta_s)^2 + (\eta_s - \eta)^2 + 2(\eta_0 - \eta_s)(\eta_s - \eta) \cos \psi]^{1/2} & \eta \leq \eta_s, \\ \eta_0 - \eta & \eta \geq \eta_s, \end{cases} \quad (11)$$

where $\eta_0 \equiv \eta(t_0)$ is the present epoch, $\eta_s \equiv \eta(t_s)$ is the epoch of scattering, and ψ is the scattering angle. The potential $\phi(r)$ defined by (10) is time independent, since $\delta\rho \propto a^{-2}$ for the growing mode [$\delta\rho/\rho \propto a \propto t^{2/3}$]. That the redshift perturbation can be written in terms of such a potential is the fundamental result of Sachs and Wolfe [27].

The first two terms contributing to the redshift perturbation (7) reflect the difference in potential between the origin and destination of the photon. The remaining terms are Doppler shifts: they arise from the peculiar velocities of the matter with respect to the unperturbed background Einstein-de Sitter cosmology. These velocities are produced by the peculiar gravitational acceleration $-d\phi/dr$ acting over arc-parameter “time” η . In our case, the peculiar velocities are radial, so the Doppler shift is proportional to $dr/d\eta$, which is the cosine of the angle between the photon momentum and the radial direction. The Doppler shift at the scattering epoch depends upon the change in that cosine, whence the $(1 - \cos \psi)$ factor. Since $\eta_{\text{rec}}/\eta_0 = (1 + z_{\text{rec}})^{-1/2} \approx 10^{-1.5} \ll 1$, we simplify Eqs. (7) and (11) by replacing η_{rec} with 0.

The CMB spectrum seen by an observer at $r = 0$ is a weighted sum of redshifted Planck functions:

$$I_\nu(r = 0) = \int B_\nu(e^{-\zeta}T_0)dP(\zeta) \quad (12)$$

where $B_\nu(T_0)$ is a Planck function at the present-day CMB temperature, $T_0 \equiv T_{\text{rec}}/(1 + z_{\text{rec},0})$, and $P(\zeta)$ is the probability distribution for ζ . The probability density $dP/d\zeta$ consists of two parts: a delta function of area $e^{-\tau_{\text{IGM}}}$ at the value of ζ for unscattered photons [computed from (7) by setting $\psi = 0$]; and a continuous part of total area $1 - e^{-\tau_{\text{IGM}}} \approx \tau_{\text{IGM}} \ll 1$ representing the once-scattered photons.

It is clear that if ζ were the same along all paths, the spectrum $I_\nu(0)$ would remain thermal but would have temperature $e^{-\zeta}T_0$ instead of T_0 . Thus nonthermal distortions

depend upon the differences in ζ among scattered paths. This can be demonstrated formally by expanding $B_\nu(T)$ in equation (12) as a Taylor series in $\ln T$ about the point $\ln T = \ln T_0 - \langle \zeta \rangle$, with the result

$$I_\nu(0) = B_\nu(T_0 e^{-\langle \zeta \rangle}) + \left[\sum_{n=2}^{\infty} \langle (\zeta - \langle \zeta \rangle)^n \rangle \left(T \frac{\partial}{\partial T} \right)^n B_\nu(T) \right]_{T_0 e^{-\langle \zeta \rangle}} \quad (13)$$

The moments of ζ are

$$\langle \zeta^n \rangle \equiv \int \zeta^n dP(\zeta). \quad (14)$$

If the width of the redshift distribution $P(\zeta)$ is small, as it is when $|\phi(r)|, |r\phi'(r)| \ll c^2$, we do not need to go beyond the second-derivative term in the expansion (13). So we have only to compute the variance in the log redshift. The unscattered paths contribute to $\langle \zeta \rangle$ but not to the variance $\langle (\zeta - \langle \zeta \rangle)^2 \rangle$. So we may compute modified moments, marked by a prime, by averaging over the scattered paths only:

$$\langle \zeta^m \rangle' = \int_{\eta_{\text{rec}}}^{\eta_0} d\eta_s \frac{d\tau}{d\eta_s} \int_{-1}^{+1} d\cos\psi \frac{3}{8} (1 + \cos^2\psi) \zeta^m(\eta_s, \psi), \quad (15)$$

where $\zeta(\eta_s, \psi)$ is the function (7).

The differential optical depth is

$$\begin{aligned} \frac{d\tau}{d\eta_s} &= \sigma_T n_e(t) c \frac{dt}{d\eta_s} = 3\sigma_T n_e(t_0) \frac{ct_0}{\eta_0} \\ &\times \begin{cases} (\eta_s/\eta_0)^{-4} & \text{if } \eta_s \geq \eta_{\text{ion}}, \\ 0 & \text{if } \eta_{\text{rec}} < \eta_s < \eta_{\text{ion}}. \end{cases} \end{aligned} \quad (16)$$

Following the discussion above, we have taken the comoving density of electrons to be constant from the present back to a redshift factor $1 + z_{\text{ion}} = (\eta_{\text{ion}}/\eta_0)^{-2}$. The total optical depth is (assuming full ionization of ^4He)

$$\begin{aligned} \tau_{\text{IGM}} &= \sigma_T n_e(t_0) ct_0 [(1 + z_{\text{ion}})^{3/2} - 1] \\ &\approx 1.0 \times 10^{-3} f_{\text{IGM}} [(1 + z_{\text{ion}})^{3/2} - 1]. \end{aligned} \quad (17)$$

To compute the variance of ζ and hence the distortion (13), we must assume a functional form for the perturbed potential $\phi(r)$. Rather arbitrarily, we choose

$$\phi(r) = \phi_0 \cos(\omega r), \quad (18)$$

in which ϕ_0 is a normalization and $2\pi/\omega$ is an adjustable comoving radial wavelength. A general spherically-symmetric linear perturbation could be written as a superposition of Fourier components of this form. To fix the meaning of ω , $a(t)$ will be scaled so that $\eta_0 = 1$ [$a(t_0) = 3t_0$]. Therefore $r = 1$ is the present horizon, and $a(t_0) = 3t_0$. After substitution of

equations (7) and (18) into the formula for the moments (15), the integrations over ψ and η_s can be expressed in closed but lengthy form, which we omit.

The variance $\langle \zeta^2 \rangle - \langle \zeta \rangle^2 \equiv \langle \Delta \zeta^2 \rangle$ is clearly proportional to ϕ_0^2 , but it vanishes for certain functional forms of $\phi(r)$. In the limit $\omega \rightarrow 0$, $\phi(r)$ becomes constant and has no effect on the observed spectrum [cf. equation (7)]. Even a quadratic potential $\phi(r) = \phi_0 \cdot [1 - (\omega r)^2/2]$ would spoil neither the thermality of $I_\nu(0)$ nor the homogeneity of the geometry: such a perturbation corresponds to a slightly non-flat Friedmann model. It follows that for $\omega \ll 1$, $\langle \zeta^2 \rangle \propto \omega^8 \phi_0^2$.

One coordinate-independent way to characterize the sensitivity of the spectrum (13) to the degree of radial inhomogeneity is to compare the spectral distortion at $r = 0$ with the *anisotropy* seen by a typical *non-central* observer. In the limit $\tau_{\text{IGM}} \rightarrow 0$, an observer at $r_{\text{obs}} > 0$ and $\eta = \eta_0$ sees the CMB as thermal in every direction, but the temperature varies with angle according to

$$\frac{\delta T}{T}(\theta) = \frac{1}{3} \left\{ \phi[r_{\text{rec}}(\theta)] - \phi(r_{\text{obs}}) + \eta_0 \cos \theta \frac{d\phi}{dr}(r_{\text{obs}}) \right\} \quad (19)$$

apart from a constant. Here θ is measured with respect to the radial direction, and

$$r_{\text{rec}}(\theta) = [r_{\text{obs}}^2 + \eta_0^2 + 2\eta_0 r_{\text{obs}} \cos \theta]^{1/2} \quad (20)$$

is the locus of the non-central observer's horizon. We calculate the variance

$$\begin{aligned} \left(\frac{\Delta T}{T} \right)_{r_{\text{obs}}}^2 &\equiv \int_{-1}^{+1} d \cos \theta \left[\frac{\delta T(\theta)}{T} \right]^2 \\ &\quad - \left[\int_{-1}^{+1} d \cos \theta \frac{\delta T(\theta)}{T} \right]^2. \end{aligned} \quad (21)$$

When $\omega \gg 1$, $(\Delta T/T)^2$ can oscillate rapidly with r_{obs} , so we compute a smoothly-tapered radial average:

$$\left\langle \left(\frac{\Delta T}{T} \right)^2 \right\rangle \equiv \int_0^1 dr_{\text{obs}} \left(\frac{\Delta T}{T} \right)_{r_{\text{obs}}}^2 \frac{1}{2} \sin^2(\pi r_{\text{obs}}). \quad (22)$$

Finally, we define the “normalized” spectral distortion \hat{y} by

$$\tau_{\text{IGM}} \hat{y} \equiv \frac{\langle \Delta \zeta^2 \rangle}{\langle (\Delta T/T)^2 \rangle} \quad (23)$$

Since both the numerator (the spectral distortion at the center of the universe) and the denominator (the typical angular temperature variance off center) are proportional to ϕ_0^2 , the quantity \hat{y} is independent of the amplitude of the potential fluctuations in the linear regime. We have also scaled the total optical depth (17) out of \hat{y} . However, \hat{y} does depend somewhat on ω and z_{ion} .

Table I shows some representative values of \hat{y} . The notation “0⁺” under the heading for $\omega/2\pi$ denotes the limit as $\omega \rightarrow 0$. One sees from the Table that \hat{y} is essentially independent

of ω if there are two or more cycles within the horizon ($\omega/2\pi \gtrsim 2$), but \hat{y} decreases sharply at smaller ω . Thus the spectral distortion is relatively insensitive to density perturbations that are well fit by a quadratic function of radius (corresponding to a quartic potential, since $\delta\rho/\rho \propto \eta^2 \nabla^2 \phi$).

The value of \hat{y} decreases noticeably if $z_{\text{tection}} \gg 1$. In that case, since the differential optical depth $d\tau \propto \sqrt{1+z}$, most of the scatterings occur very early, at $z \sim z_{\text{ion}}$, when the photons have not moved far from their positions at recombination. If ω is moderate, the potential ϕ is nearly constant over this small range in r_i . On the other hand, if $\omega \gg 1$, the redshift perturbation (7) is dominated by the Doppler term $\eta_s d\phi/d\eta_s$, which decreases with scattering epoch $\eta_s \propto (1+z_s)^{-1/2}$. [For $\omega \gg 1$, the denominator of (23) is also dominated by Doppler shifts—those of the noncentral observers themselves—but these shifts are evaluated at the present epoch and are not suppressed by the $(1+z_s)^{-1/2}$ factor.] So \hat{y} is small for large z_{ion} , regardless of ω .

We have scaled the total optical depth τ_{IGM} out of \hat{y} , however, so that the observed distortion is proportional to $\tau_{\text{IGM}}\hat{y}$. For $z_{\text{ion}} = 2, 4$, and 10 , equation (17) predicts $\tau_{\text{IGM}} \approx 0.004f, 0.01f$, and $0.04f$, respectively; at larger z_{ion} , the approximation of single scattering begins to break down. At any rate, for a fixed nonzero amplitude ϕ_0 and for $\omega > 0$, the observed spectral distortion tends to increase with z_{ion} despite the decrease in \hat{y} .

We have adopted the notation “ \hat{y} ” because the spectral distortion produced by radial inhomogeneity has the same form as the distortion arising from inverse compton scattering provided that $T_\gamma \ll T_e \ll m_e c^2/k_B$, where T_γ and T_e are the photon and electron temperatures. Quite generally, a mixture of Planck functions at slightly different temperatures is indistinguishable from a slightly comptonized spectrum ([28]). The correspondance in Eq. (13) is $\langle \zeta \rangle \rightarrow -3y$ and $\langle \Delta \zeta^2 \rangle \rightarrow y$; higher-order moments of ζ are negligible if $y \ll 1$.

This means that we can translate published upper limits on comptonization of the CMB into limits on radial inhomogeneity. The limit reported by the COBE collaboration is $y < 2.5 \times 10^{-5}$ ([29]). Hence the limit on inhomogeneity as defined by (22) is

$$\begin{aligned} \overline{\left(\frac{\Delta T}{T}\right)} &\leq \left(\frac{y}{\tau_{\text{IGM}}\hat{y}(\omega, z_{\text{ion}})}\right)^{1/2} \\ &\lesssim 0.05f^{-1/2}. \end{aligned} \tag{24}$$

In the final numerical form, we have taken $\hat{y} \approx 1$ as a typical value from Table 1, and we have assumed $z_{\text{ion}} = 4$. As Table I shows, \hat{y} is much smaller and our limit correspondingly weaker if $\omega < \pi$.

IV. DISCUSSION

We have discussed three tests of the large-scale radial homogeneity of the universe. All three involve possible distortions or variations in the CMB spectrum. There is, however, a fourth test that is more widely recognized than any of these: big-bang nucleosynthesis of the light elements (henceforth BBN).

The standard theory of BBN predicts the primordial abundances of the light elements ^1H , ^2H , ^3He , ^4He , and ^7Li in terms of a single parameter, η , the number of baryons per CMB

photon. (We ignore complications such as a nonstandard number of neutrino flavors.) From the observed *relative* abundances of these elements in the local universe, it appears that [22]

$$2.8 \leq \eta_{10} \leq 4.0, \quad \eta_{10} \equiv 10^{10} \eta. \quad (25)$$

Measuring abundance ratios such as $N(^2\text{H})/N(^1\text{H})$ in cosmologically distant systems tests radial homogeneity without reference to the CMB or to BBN, since it is sensitive to inhomogeneities in whatever processes create and destroy these elements. Since such measurements have only just begun to come in during the past year [30], and since a consensus has not yet been reached, this is not the fourth test we referred to above. Nevertheless it may give very interesting results in the near future.

The value of η constrained by (25) pertains to photons that resided here at the time of nucleosynthesis. Those photons are no longer with us: they have been streaming away since recombination and are now almost at the horizon. On the hypothesis of homogeneity, however, the CMB photons seen today stand as proxies for those long-gone photons. In particular, we may evaluate η using the present-day density of CMB photons in our vicinity, $n_\gamma \propto T_{\text{CMB}}^3$. Hence the local mean density of baryons should be [22]

$$\bar{n}_B = 20.3 T_{\text{CMB}}^3 \eta \text{ cm}^{-3} \approx (1.4 \pm 0.3) \times 10^{-7} \text{ cm}^{-3}, \quad (26)$$

where we have taken $T = 2.726 \text{ K}$ ([29]) and adopted the range (25) for η . To the extent that \bar{n}_B can be measured and compared with this prediction, one tests the *spatial* constancy of η . Under the assumptions (S.1)-(S.5) of Sec. III, we have

$$(n_B^{\text{meas.}}/n_B^{\text{pred.}}) - 1 = \phi(0) - \phi(r = \eta_{\text{rec}}), \quad (27)$$

provided both sides of this equation are small compared to unity. To be competitive with the limits presented in Secs. II-III, one should measure \bar{n}_B to $\sim 30\%$ or better.

It seems that \bar{n}_B has not yet been measured to the required accuracy. It is estimated that the luminous parts of galaxies account for a mean mass density $\approx 5 \times 10^{-8} e^{\pm 0.3} h^2 m_H \text{ cm}^{-3}$, where $h \equiv H_0/(100 \text{ km s}^{-1} \text{ Mpc}^{-1})$, based on the observed luminosity density and an assumed mass-to-light ratio $M/L_B = 12 M_{\text{sun}}/L_{\text{sun}}$ [20]. Since this ratio counts all of the mass within the Holmberg radius, it may include some nonbaryonic dark matter; and probably $h < 1$. Thus the visible parts of galaxies fall short of the nucleosynthetic prediction (27) by a factor of at least 3. In some clusters of galaxies, hot X-ray-emitting gas increases the total baryonic M/L_B by a factor 5.6 to 16, depending on h [23]. In summary, while the observed baryon density is perhaps consistent with Eq. (26), the bookkeeping is not yet accurate enough to yield a 10% limit on the radial homogeneity of the CMB temperature.

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REFERENCES

- * Electronic address: `jeremy@astro.Princeton.EDU`
- [1] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Spacetime* (Cambridge University Press, Cambridge, 1973).
 - [2] R. C. Tolman, Proc. Nat. Acad. Sci. USA **20**, 169 (1934); H. Bondi, Mon. Not. R. Astron. Soc, **107**, 410 (1947).
 - [3] A. Linde, D. Linde, and A. Mezhlumian, 1994, SISSA e-print hep-th/9411111 (1994).
 - [4] R. H. Becker, R. L. White, and A. L. Edwards, Astrophys. J., Suppl. Ser. **75**, 1 (1991); P. C. Gregory and J. J. Condon, Astrophys. J., Suppl. Ser. **75**, 1011 (1991).
 - [5] M. A. Strauss, M. Davis, A. Yahil, and J. P. Huchra, Astrophys. J. **385**, 421 (1992).
 - [6] C. L. Bennett, *et al.* 1994, Astrophys. J. **436**, 423 (1994).
 - [7] D. C. Koo and R. G. Kron, Annu. Rev. Astron. Astrophys. **30**, 613 (1992).
 - [8] S. J. Lilly, L. L. Cowie, and J. P. Gardner, Astrophys. J. **369**, 79 (1991).
 - [9] L. L. Cowie, J. P. Gardner, S. J. Lilly, & I. Mclean, Astrophys. J. **360**, L1 (1990).
 - [10] D. Branch, and G. A. Tammann, Annu. Rev. Astron. Astrophys. **30**, 359 (1992); R. Phillips, M. Astrophys. J. Lett. **413**, L105 (1993); A. G. Riess, W. H. Press, and R. P. Kirshner, Astrophys. J. Lett., in press (1995).
 - [11] R. Narayan, Astrophys. J. Lett. **378**, L5 (1991).
 - [12] R. A. Sunyaev, and Ya. B. Zel'dovich, Comments Astrophys. Space Phys. **4**, 173 (1972).
 - [13] R. A. Sunyaev, and Ya. B. Zel'dovich, Mon. Not. R. Astron. Soc. **190**, 413 (1980).
 - [14] U. Klein, R. Schlickeiser, R. Wielebinski, and Y. Rephaeli, Y. Astron. Astrophys. **244**, 43 (1991); T. Herbig, A. C. S. Readhead, and C. R. Lawrence, Bull. Am. Astron. Soc. **24**, 1263 (1992); M. Jones *et al.*, Nature (London), **365**, 320 (1993). Older measurements are reviewed by M. Birkinshaw, in *The cosmic microwave background: 25 years later*, edited by N. Mandolesi and N. Vittorio (Kluwer, Dordrecht, 1990), p. 77.
 - [15] T. M. Wilbanks *et al.*, Astrophys. J. Lett. **427**, L75 (1994); P. Andreani, P. *et al.*, The Messenger, **78**, 41 (1994).
 - [16] I. M. McHardy *et al.*, Mon. Not. R. Astron. Soc. **242**, 215 (1990); M. Birkinshaw, J. P. Hughes, and K. A. Arnaud, Astrophys. J. **379**, 466 (1991); M. Birkinshaw, and J. P. Hughes, 1994, Astrophys. J. **420**, 33 (1994).
 - [17] W. L. Freedman *et al.*, Nature (London) **371**, 757 (1994).
 - [18] J. N. Bahcall, and R. A. Wolfe, Astrophys. J., **152**, 701 (1968); P. Thaddeus, Annu. Rev. Astron. Astrophys. **10**, 305 (1972).
 - [19] A. Songaila, *et al.*, Nature, **371**, 43 (1994).
 - [20] P. J. E. Peebles, *Principles of Physical Cosmology* (Princeton Univ. Press, Princeton, 1993).
 - [21] W. H. Press and G. B. Rybicki, Astrophys. J. **418**, 585 (1993).
 - [22] T. P. Walker, G. Steigman, D. N. Schramm, K. A. Olive, and H. S. Kang, Astrophys. J. **376**, 51 (1991).
 - [23] S. D. M. White, J. F. Navarro, A. E. Evrard, and C. S. Frenk, Nature (London) **366**, 429 (1993).
 - [24] R. Fusco-Femiano and J. P. Hughes, Astrophys. J. **429**, 545 (1994); S. D. M. White *et al.* Mon. Not. R. Astron. Soc. **269**, 589 (1994).

- [25] D. H. Weinberg, A. Szomoru, P. Guhathakurta, and J. H. van Gorkom, *Astrophys. J.* **372**, L13 (1991); J. N. Bahcall *et al.* **397**, 68 (1992).
- [26] M. White, D. Scott, and J. Silk, *Annu. Rev. Astron. Astrophys.* **30**, 613 (1994).
- [27] R. K. Sachs and A. M. Wolfe, *Astrophys. J.* **147**, 73 (1967).
- [28] R. Daly, *Astrophys. J.* **371**, 14 (1991).
- [29] J. G. Mather *et al.*, *Astrophys. J.* **420**, 439 (1994).
- [30] A. Songaila, L. L. Cowie, C. J. Hogan, and M. Rugers, *Nature (London)* **368**, 599; R. F. Carswell, M. Rauch, R. J. Weymann, A. J. Cooke, and J. K. Webb, *Mon. Not. R. Astron. Soc.* **268**, L1 (1994); D. Tytler, private communication (1995).

TABLES

TABLE I. Normalized spectral distortion \hat{y} .

$\omega/2\pi$	$z_{\text{ion}} = 2$	$z_{\text{ion}} = 4$	$z_{\text{ion}} = 100$
0^+	0.040	0.029	0.0027
0.5	0.16	0.11	0.0076
1.0	1.31	0.98	0.026
2.0	2.31	1.23	0.056
10.	2.18	1.49	0.10
10^3	2.20	1.53	0.11